

Symmetry classification and conservation laws for higher order Camassa-Holm equation

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Abstract

Lie symmetry group method is applied to study for the higher order Camassa-Holm equation. The symmetry group and its optimal system are given. Furthermore, preliminary classification of its group invariant solutions, symmetry reduction and nonclassical symmetries are investigated. Finally conservation laws for the higher order Camassa-Holm equation are presented.

Key words: Lie symmetry, Group-invariant solutions, Higher order Camassa-Holm equation, Optimal system, Conservation laws.

1 Introduction

In the study of shallow water waves, Camassa and Holm [7] derived a nonlinear dispersive shallow water wave equation

$$u_t - u_{x^2t} + 3uu_x = 2u_xu_{x^2} + uu_{x^3}, \quad (1)$$

which is called Camassa-Holm equation (CH). Here $u(x, t)$ denotes the fluid velocity at time t in the x direction or, equivalently, the high of water's free surface above a flat bottom. Eq. (1) has a bi-Hamiltonian structure [13,17], and is completely integrable [7,11]. It has many conservation laws [21]. Moreover, the CH equation is a re-expression of geodesic flow on the diffeomorphism group of the line. Holm, Marsden and Ratiu [14] have shown that CH equation in n dimensions describes geodesic motion on the diffeomorphism group of R^n with respect to metric given by the H^1 norm or Euclidean fluid velocity. Misiolek [19] has shown that the CH equation represents a geodesic flow on the Bott-Virasoro group. Kouranbaeva [15] has shown that the CH equation (for the case $k = 0$) is a geodesic spray of the weak Riemannian metric on the diffeomorphism group of the line or the circle obtained by the right translation of the H^1 inner product over the entire group. This equation admits well known properties and a rich literature is devoted to it.

In recent years, many researchers have been researched on the Camassa-Holm equation. They extend the studies to the generalized CH equation, higher order CH equations and so on. Lixin Tian, Chunyu Shen and Danping Ding gave the optimal control of the viscous CH equation under the boundary condition and proved the existence and uniqueness of optimal solution to the viscous CH equation in a short interval (see [26]). Using geometrical methods, higher order CH equations have been treated in [12]. The well-posedness of higher order CH equations were considered in [10].

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The formulation of the higher order Camassa-Holm equation which was recently derived by Coclite, Holden and Karlsen in [10] is

$$\begin{aligned} B_k(u, u) &:= A_k^{-1} C_k(u) - uu_x, \\ A_k(u) &:= \sum_{j=0}^k (-1)^j \partial_x^2 j u, \\ C_k(u) &= -u A_k(\partial_x u) + A_k(u \partial_x u) - 2 \partial_x u A_k(u). \end{aligned} \tag{2}$$

where k is a positive integer. In cases $k = 0$ and $k = 1$, Eq. (2) becomes the inviscid Burgers equation and the Camassa-Holm equation respectively.

In this paper we only consider the case $k = 2$ of Eq. (2). It also can be rewritten as

$$\Delta := u_t - u_{x^2 t} + u_{x^4 t} + 3uu_x - 2u_x u_{x^2} - uu_{x^3} + 2u_x u_{x^4} + uu_{x^5} = 0. \tag{3}$$

The theory of Lie symmetry groups of differential equations was developed by Sophus Lie [16], which was called classical Lie method. Nowadays, application of Lie transformations group theory for constructing the solutions of nonlinear partial differential equations (PDEs) can be regarded as one of the most active fields of research in the theory of nonlinear PDEs and applications. Such Lie groups are invertible point transformations of both the dependent and independent variables of the differential equations. The symmetry group methods provide an ultimate arsenal for analysis of differential equations and is of great importance to understand and to construct solutions of differential equations. Several applications of Lie groups in the theory of differential equations were discussed in the literature, the most important ones are: reduction of order of ordinary differential equations, construction of invariant solutions, mapping solutions to other solutions and the detection of linearizing transformations. For many other applications of Lie symmetries see [22,6,4].

The fact that symmetry reductions for many PDEs are unobtainable by applying the classical symmetry method, motivated the creation of several generalizations of the classical Lie group method for symmetry reductions. The nonclassical symmetry method of reduction was devised originally by Bluman and Cole in 1969 [5], to find new exact solutions of the heat equation. The description of the method is presented in [8,18]. Many authors have used the nonclassical method to solve PDEs. In [9] Clarkson and Mansfield have proposed an algorithm for calculating the determining equations associated to the nonclassical method. A new procedure for finding nonclassical symmetries has been proposed by Bilă and Niesen in [1].

Many PDEs in the applied sciences and engineering are continuity equations which express conservation of mass, momentum, energy, or electric charge. Such equations occur in, e.g., fluid mechanics, particle and quantum physics, plasma physics, elasticity, gas dynamics, electromagnetism, magneto-hydro-dynamics, nonlinear optics, etc. In the study of PDEs, conservation laws are important for investigating integrability and linearization mappings and for establishing existence and uniqueness of solutions. They are also used in the analysis of stability and global behavior of solutions [2,3,24,25].

This work is organized as follows. In section 2 we recall some results needed to construct Lie point symmetries of a given system of differential equations. In section 3, we give the general form of a infinitesimal generator admitted by equation (3) and find transformed solutions. Section 4, is devoted to the nonclassical symmetries of the higher order CH model, symmetries generated when a supplementary condition, the invariance surface condition, is imposed. In Section 5, we construct the optimal system of one-dimensional subalgebras. Lie invariants, similarity reduced equations and differential invariants corresponding to the infinitesimal symmetries of equation (3) are obtained in section 6. Finally in last section, the conservation laws of the equation (3) are obtained.

2 Method of Lie Symmetries

In this section, we recall the general procedure for determining symmetries for any system of partial differential equations see [22,20,6,4]. To begin, let us consider the general case of a nonlinear system E of partial differential equations of order n in p independent and q dependent variables is given as a system of equations

$$\Delta_\nu(x, u^{(n)}) = 0, \quad \nu = 1, \dots, l, \tag{4}$$

involving $x = (x^1, \dots, x^p)$, $u = (u^1, \dots, u^q)$ and the derivatives of u with respect to x up to n , where $u^{(n)}$ represents all the derivatives of u of all orders from 0 to n . We consider a one-parameter Lie group of infinitesimal transformations acting on the independent and dependent variables of the system (4)

$$\begin{aligned}\tilde{x}^i &= x^i + s\xi^i(x, u) + O(s^2), & i &= 1 \dots, p, \\ \tilde{u}^j &= u^j + s\varphi^j(x, u) + O(s^2), & j &= 1 \dots, q,\end{aligned}\tag{5}$$

where s is the parameter of the transformation and ξ^i, η^j are the infinitesimals of the transformations for the independent and dependent variables, respectively. The infinitesimal generator \mathbf{v} associated with the above group of transformations can be written as

$$\mathbf{v} = \sum_{i=1}^p \xi^i(x, u) \partial_{x^i} + \sum_{\alpha=1}^q \varphi^\alpha(x, u) \partial_{u^\alpha}.\tag{6}$$

A symmetry of a differential equation is a transformation which maps solutions of the equation to other solutions. The invariance of the system (4) under the infinitesimal transformations leads to the invariance conditions (Theorem 2.36 of [22])

$$\text{Pr}^{(n)}\mathbf{v}[\Delta_\nu(x, u^{(n)})] = 0, \quad \nu = 1, \dots, l, \quad \text{whenever} \quad \Delta_\nu(x, u^{(n)}) = 0,\tag{7}$$

where $\text{Pr}^{(n)}$ is called the n^{th} order prolongation of the infinitesimal generator given by

$$\text{Pr}^{(n)}\mathbf{v} = \mathbf{v} + \sum_{\alpha=1}^q \sum_J \varphi_\alpha^J(x, u^{(n)}) \partial_{u_J^\alpha},\tag{8}$$

where $J = (j_1, \dots, j_k)$, $1 \leq j_k \leq p$, $1 \leq k \leq n$ and the sum is over all J 's of order $0 < \#J \leq n$. If $\#J = k$, the coefficient φ_α^J of $\partial_{u_J^\alpha}$ will only depend on k -th and lower order derivatives of u , and

$$\varphi_\alpha^J(x, u^{(n)}) = D_J(\varphi_\alpha - \sum_{i=1}^p \xi^i u_i^\alpha) + \sum_{i=1}^p \xi^i u_{J,i}^\alpha,\tag{9}$$

where $u_i^\alpha := \partial u^\alpha / \partial x^i$ and $u_{J,i}^\alpha := \partial u_J^\alpha / \partial x^i$.

One of the most important properties of these infinitesimal symmetries is that they form a Lie algebra under the usual Lie bracket.

3 Lie symmetries for the higher order CH equation

We consider the one parameter Lie group of infinitesimal transformations on $(x^1 = x, x^2 = t, u^1 = u)$,

$$\begin{aligned}\tilde{x} &= x + s\xi(x, t, u) + O(s^2), \\ \tilde{t} &= t + s\eta(x, t, u) + O(s^2), \\ \tilde{u} &= u + s\varphi(x, t, u) + O(s^2),\end{aligned}\tag{10}$$

where s is the group parameter and $\xi^1 = \xi$, $\xi^2 = \eta$ and $\varphi^1 = \varphi$ are the infinitesimals of the transformations for the independent and dependent variables, respectively. The associated vector field is of the form:

$$\mathbf{v} = \xi(x, t, u) \partial_x + \eta(x, t, u) \partial_t + \varphi(x, t, u) \partial_u.\tag{11}$$

and, by (8) its fifth prolongation is

$$\begin{aligned}\text{Pr}^{(5)}\mathbf{v} &= \mathbf{v} + \varphi^x \partial_{u_x} + \varphi^t \partial_{u_t} + \varphi^{x^2} \partial_{u_{x^2}} + \varphi^{xt} \partial_{u_{xt}} + \varphi^{t^2} \partial_{u_{t^2}} \\ &\quad + \varphi^{x^3} \partial_{u_{x^3}} + \varphi^{x^2 t} \partial_{u_{x^2 t}} + \varphi^{xt^2} \partial_{u_{xt^2}} + \varphi^{t^3} \partial_{u_{t^3}} + \dots + \varphi^{xt^4} \partial_{u_{xt^4}} + \varphi^{t^5} \partial_{u_{t^5}}.\end{aligned}\tag{12}$$

where, for instance by (9) we have

$$\begin{aligned}\varphi^x &= D_x(\varphi - \xi u_x - \eta u_t) + \xi u_{x^2} + \eta u_{xt}, \\ \varphi^t &= D_t(\varphi - \xi u_x - \eta u_t) + \xi u_{xt} + \eta u_{t^2}, \\ &\vdots \\ \varphi^{t^5} &= D_x^5(\varphi - \xi u_x - \eta u_t) + \xi u_{x^5 t} + \eta u_{t^5},\end{aligned}\tag{13}$$

where D_x and D_t are the total derivatives with respect to x and t respectively. By (7) the vector field \mathbf{v} generates a one parameter symmetry group of the Eq. (3) if and only if

$$\text{Pr}^{(5)}\mathbf{v}[u_t - u_{x^2 t} + u_{x^4 t} + 3uu_x - 2u_x u_{x^2} - uu_{x^3} + 2u_x u_{x^4} + uu_{x^5}] = 0, \quad \text{whenever} \quad \Delta = 0.\tag{14}$$

The condition (14) is equivalent to

$$\begin{aligned}(3u_x - u_{x^3} + u_{x^5})\varphi + \varphi^t + (3u - 2u_{x^2} + 2u_{x^4})\varphi^x - 2u_x \varphi^{x^2} - u\varphi^{x^3} - \varphi^{x^2 t} + 2u_x \varphi^{x^4} + u\varphi^{x^5} + \varphi^{x^4 t} &= 0, \\ \text{whenever} \quad u_t - u_{x^2 t} + u_{x^4 t} + 3uu_x - 2u_x u_{x^2} - uu_{x^3} + 2u_x u_{x^4} + uu_{x^5} &= 0.\end{aligned}\tag{15}$$

Substituting (13) into (15), and equating the coefficients of the various monomials in partial derivatives with respect to x and various power of u , we can find the determining equations for the symmetry group of the Eq. (3). Solving this equations, we get the following forms of the coefficient functions

$$\xi = c_3, \quad \eta = c_1 t + c_2, \quad \varphi = -c_1 u.\tag{16}$$

where c_1 , c_2 and c_3 are arbitrary constant. Thus, the Lie algebra \mathfrak{g} of infinitesimal symmetry of the Eq. (3) is spanned by the three vector fields

$$\mathbf{v}_1 = \partial_x, \quad \mathbf{v}_2 = \partial_t, \quad \mathbf{v}_3 = t\partial_t - u\partial_u.\tag{17}$$

The commutation relations between these vector fields are given in the Table 1. The Lie algebra \mathfrak{g} is solvable, because if $\mathfrak{g}^{(1)} = \langle \mathbf{v}_i, [\mathbf{v}_i, \mathbf{v}_j] \rangle = [\mathfrak{g}, \mathfrak{g}]$, we have $\mathfrak{g}^{(1)} = \langle \mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3 \rangle$, and $\mathfrak{g}^{(2)} = [\mathfrak{g}^{(1)}, \mathfrak{g}^{(1)}] = \langle \mathbf{v}_2 \rangle$, so, we have a chain of ideals $\mathfrak{g}^{(1)} \supset \mathfrak{g}^{(2)} \supset 0$.

Table 1

<i>The commutator table</i>			
$[\mathbf{v}_i, \mathbf{v}_j]$	\mathbf{v}_1	\mathbf{v}_2	\mathbf{v}_3
\mathbf{v}_1	0	0	0
\mathbf{v}_2	0	0	\mathbf{v}_2
\mathbf{v}_3	0	$-\mathbf{v}_2$	0

To obtain the group transformation which is generated by the infinitesimal generators \mathbf{v}_i for $i = 1, 2, 3$ we need to solve the three systems of first order ordinary differential equations

$$\begin{aligned}\frac{d\tilde{x}(s)}{ds} &= \xi_i(\tilde{x}(s), \tilde{t}(s), \tilde{u}(s)), \quad \tilde{x}(0) = x, \\ \frac{d\tilde{t}(s)}{ds} &= \eta_i(\tilde{x}(s), \tilde{t}(s), \tilde{u}(s)), \quad \tilde{t}(0) = t, \quad i = 1, 2, 3 \\ \frac{d\tilde{u}(s)}{ds} &= \varphi_i(\tilde{x}(s), \tilde{t}(s), \tilde{u}(s)), \quad \tilde{u}(0) = u.\end{aligned}$$

Exponentiating the infinitesimal symmetries of equation (3), we get the one-parameter groups $G_i(s)$ generated by

\mathbf{v}_i for $i = 1, 2, 3$

$$\begin{aligned} G_1 : (t, x, u) &\longmapsto (x + s, t, u), \\ G_2 : (t, x, u) &\longmapsto (x, t + s, u), \\ G_3 : (t, x, u) &\longmapsto (x, e^s t, e^{-s} u). \end{aligned} \tag{18}$$

Consequently,

Theorem 3.1 *If $u = f(x, t)$ is a solution of higher order CH equation, so are the functions*

$$\begin{aligned} G_1(s) \cdot f(x, t) &= f(x - s, t), \\ G_2(s) \cdot f(x, t) &= f(x, t - s), \\ G_3(s) \cdot f(x, t) &= f(x, te^{-s})e^{-s}. \end{aligned} \tag{19}$$

4 Nonclassical symmetries for the higher order CH equation

In this section we would like to apply the nonclassical method to the higher order CH equation. The graph of a solution

$$u^\alpha = f^\alpha(x_1, \dots, x_p), \quad \alpha = 1, \dots, q \tag{20}$$

to the system (4) defines an p -dimensional submanifold $\Gamma_f \subset \mathbf{R}^p \times \mathbf{R}^q$ of the space of independent and dependent variables. The solution will be invariant under the one-parameter subgroup generated by vector (6) if and only if Γ_f is an invariant submanifold of this group. By applying the well known criterion of invariance of a submanifold under a vector field we get that (20) is invariant under vector (6) if and only if f satisfies the first order system E_Q of partial differential equations

$$Q^\alpha(x, u, u^{(1)}) = \varphi^{(\alpha)}(x, u) - \sum_{i=1}^p \xi^i(x, u) u_i^\alpha = 0, \quad \alpha = 1, \dots, q \tag{21}$$

known as the invariant surface conditions. The q -tuple $Q = (Q^1, \dots, Q^q)$ is known as the characteristic of the vector field (6). In what follows, the n -th prolongation of the invariant surface conditions (21) will be denoted by $E_Q^{(n)}$, which is a n -th order system of partial differential equations obtained by appending to (21) its partial derivatives with respect to the independent variables of orders $j \leq n - 1$.

For the system (4), (21) to be compatible, the n -th prolongation $\text{Pr}^{(n)}\mathbf{v}$ of the vector field \mathbf{v} must be tangent to the intersection $E \cap E_Q^{(n)}$

$$\text{Pr}^{(n)}\mathbf{v}(\Delta_\nu)|_{E \cap E_Q^{(n)}} = 0, \quad \nu = 1, \dots, l. \tag{22}$$

If the equations (22) are satisfied, then the vector field (6) is called a nonclassical infinitesimal symmetry of the system (4). The relations (22) are generalizations of the relations (7) for the vector fields of the infinitesimal classical symmetries. A similar procedure is applicable to the case of the nonclassical infinitesimal symmetries with an evident difference that in general one has fewer determining equations than in the classical case. Therefore, we expect that nonclassical symmetries are much more numerous than classical ones, since any classical symmetry is clearly a nonclassical one. The important feature of determining equations for nonclassical symmetries is that they are nonlinear, this implies that the space of nonclassical symmetries does not, in general, form a vector space. For more theoretical background see [23,1].

If we assume that the coefficient of ∂_t of the vector field (6) does not identically equal zero, then for the vector field

$$\mathbf{v} = \xi(x, t, u)\partial_x + \partial_t + \varphi(x, t, u)\partial_u \tag{23}$$

the invariant surface conditions are

$$u_t + \xi u_x = \varphi, \quad (24)$$

Calculating equations (22) and inserting φ from (24) in to it, we can find the determining equations by equating the coefficients of the various monomials in partial derivatives with respect to x and various power of u . Solving this equations, we get $\xi = c$ and $\varphi = 0$, where c is arbitrary constant.

Now assume that the coefficient of ∂_t in (23) equals zero and try to find the infinitesimal nonclassical symmetries of the form

$$\mathbf{v} = \partial_x + \varphi(x, t, u)\partial_u \quad (25)$$

for which the invariant surface conditions is $u_x = \varphi$. Similar the previous case, we can find determining equations. Solving this equations, we get $\varphi = 0$. This means that no supplementary symmetries, of non-classical type, are specific for our models.

5 Optimal system for the higher order CH equation

In general, to each s -parameter subgroup H of the full symmetry group G of a system of differential equations in $p > s$ independent variables, there will correspond a family of group-invariant solutions. Since there are almost always an infinite number of such subgroups, it is not usually feasible to list all possible group-invariant solutions to the system. We need an effective, systematic means of classifying these solutions, leading to an "optimal system" of group-invariant solutions from which every other such solution can be derived.

Definition 5.1 Let G be a Lie group with Lie algebra \mathfrak{g} . An optimal system of s -parameter subgroups is a list of conjugacy inequivalent s -parameter subalgebras with the property that any other subgroup is conjugate to precisely one subgroup in the list. Similarly, a list of s -parameter subalgebras forms an optimal system if every s -parameter subalgebra of \mathfrak{g} is equivalent to a unique member of the list under some element of the adjoint representation: $\bar{\mathfrak{h}} = \text{Ad}(g(\mathfrak{h}))$, $g \in G$. [22]

Theorem 5.2 Let H and \bar{H} be connected s -dimensional Lie subgroups of the Lie group G with corresponding Lie subalgebras \mathfrak{h} and $\bar{\mathfrak{h}}$ of the Lie algebra \mathfrak{g} of G . Then $\bar{H} = gHg^{-1}$ are conjugate subgroups if and only if $\bar{\mathfrak{h}} = \text{Ad}(g(\mathfrak{h}))$ are conjugate subalgebras. (Proposition 3.7 of [22])

By theorem (5.2), the problem of finding an optimal system of subgroups is equivalent to that of finding an optimal system of subalgebras. For one-dimensional subalgebras, this classification problem is essentially the same as the problem of classifying the orbits of the adjoint representation, since each one-dimensional subalgebra is determined by nonzero vector in \mathfrak{g} . This problem is attacked by the naïve approach of taking a general element \mathbf{V} in \mathfrak{g} and subjecting it to various adjoint transformation so as to "simplify" it as much as possible. Thus we will deal with the construction of the optimal system of subalgebras of \mathfrak{g} . To compute the adjoint representation, we use the Lie series

$$\text{Ad}(\exp(\varepsilon \mathbf{v}_i) \mathbf{v}_j) = \mathbf{v}_j - \varepsilon [\mathbf{v}_i, \mathbf{v}_j] + \frac{\varepsilon^2}{2} [\mathbf{v}_i, [\mathbf{v}_i, \mathbf{v}_j]] - \cdots, \quad (26)$$

where $[\mathbf{v}_i, \mathbf{v}_j]$ is the commutator for the Lie algebra, ε is a parameter, and $i, j = 1, 2, 3$. Then we have the Table 2.

Table 2
Adjoint representation table

$\text{Ad}(\exp(\varepsilon \mathbf{v}_i) \mathbf{v}_j)$	\mathbf{v}_1	\mathbf{v}_2	\mathbf{v}_3
\mathbf{v}_1	\mathbf{v}_1	\mathbf{v}_2	\mathbf{v}_3
\mathbf{v}_2	\mathbf{v}_1	\mathbf{v}_2	$\mathbf{v}_3 - \varepsilon \mathbf{v}_2$
\mathbf{v}_3	\mathbf{v}_1	$\mathbf{v}_2 + \varepsilon \mathbf{v}_3$	\mathbf{v}_3

Theorem 5.3 An optimal system of one-dimensional Lie algebras of the higher order CH equation is provided by
(1) $\alpha \mathbf{v}_1 + \mathbf{v}_3$, (2) $\beta \mathbf{v}_1 + \mathbf{v}_2$

proof: Consider the symmetry algebra \mathfrak{g} of the Eq. (3) whose adjoint representation was determined in table 2 and let $F_i^s : \mathfrak{g} \rightarrow \mathfrak{g}$ defined by $\mathbf{v} \mapsto \text{Ad}(\exp(\varepsilon \mathbf{v}_i))\mathbf{v}$ is a linear map, for $i = 1, 2, 3$. The matrices M_i^ε of F_i^ε , $i = 1, 2, 3$, with respect to basis $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ are

$$M_1^\varepsilon = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad M_2^\varepsilon = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & \varepsilon \\ 0 & 0 & 1 \end{pmatrix}, \quad M_3^\varepsilon = \begin{pmatrix} 1 & 0 & 0 \\ 0 & e^{-\varepsilon} & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Let $\mathbf{V} = \sum_{i=1}^3 a_i \mathbf{v}_i$ is a nonzero vector field in \mathfrak{g} . We will simplify as many of the coefficients a_i as possible by acting these matrices on a vector field \mathbf{V} alternatively.

Suppose first that $a_3 \neq 0$, scaling \mathbf{V} if necessary we can assume that $a_3 = 1$, then we can make the coefficients of \mathbf{v}_2 vanish by M_2^ε , and \mathbf{V} reduced to case 1.

If $a_3 = 0$ and $a_2 \neq 0$, then we can not make vanish the coefficients of \mathbf{v}_1 and \mathbf{v}_2 by acting any matrices M_i^ε . Scaling \mathbf{V} if necessary, we can assume that $a_2 = 1$ and \mathbf{V} reduced to case 2. \square

6 Symmetry reduction and differential invariants for the higher order CH equation

Lie-group method is applicable to both linear and non-linear partial differential equations, which leads to similarity variables that may be used to reduce the number of independent variables in partial differential equations. By determining the transformation group under which a given partial differential equation is invariant, we can obtain information about the invariants and symmetries of that equation.

Symmetry group method will be applied to the (3) to be connected directly to some order differential equations. To do this, a particular linear combinations of infinitesimals are considered and their corresponding invariants are determined. The equation (3) is expressed in the coordinates (x, t, u) , so to reduce this equation is to search for its form in specific coordinates. Those coordinates will be constructed by searching for independent invariants (y, v) corresponding to the infinitesimal generator. So using the chain rule, the expression of the equation in the new coordinate allows us to the reduced equation. Here we will obtain some invariant solutions with respect to symmetries. First we obtain the similarity variables for each term of the Lie algebra \mathfrak{g} , then we use this method to reduced the PDE and find the invariant solutions.

We can now compute the invariants associated with the symmetry operators, they can be obtained by integrating the characteristic equations. For example for the operator $\alpha \mathbf{v}_1 + \mathbf{v}_2 = \alpha \partial_x + \partial_t$ characteristic equation is

$$\frac{dx}{\alpha} = \frac{dt}{1} = \frac{du}{0}. \quad (27)$$

The corresponding invariants are $y = x - \alpha t$, $v = u$ therefore, a solution of our equation in this case is $u = v(y)$. The derivatives of u are given in terms of y and v as

$$u_t = -\alpha v_y, \quad u_{x^2 t} = -\alpha v_{y^3}, \quad u_{x^4 t} = -\alpha v_{y^5}, \quad u_x = v_y, \quad u_{x^2} = v_{y^2}, \dots, u_{x^5} = v_{y^5}. \quad (28)$$

Substituting (28) into the Eq. (3), we obtain the ordinary differential equation

$$-\alpha v_y + \alpha v_{y^3} - \alpha v_{y^5} + 3v v_y - 2v_y v_{y^2} - v v_{y^3} + 2v_y v_{y^4} + v v_{y^5} = 0. \quad (29)$$

All results are coming in the tables 3 and 4.

Differential invariants help us to find general systems of differential equations which admit a prescribed symmetry group. One say, if G is a symmetry group for a system of PDEs with functionally differential invariants, then, the system can be rewritten in terms of differential invariants. For finding the differential invariants of the equation (3) up to order 2, we should solve the following systems of PDEs:

$$\frac{\partial I}{\partial x}, \quad \frac{\partial I}{\partial t}, \quad t \frac{\partial I}{\partial t} - u \frac{\partial I}{\partial u}, \quad (30)$$

Table 3
Reduction of Eq. (3)

operator	y	v	u
\mathbf{v}_1	t	u	$v(y)$
\mathbf{v}_2	x	u	$v(y)$
\mathbf{v}_3	x	$t u$	$\frac{1}{t} v(y)$
$\alpha \mathbf{v}_1 + \mathbf{v}_3$	$x - \log(t)$	$t u$	$\frac{1}{t} v(y)$
$\alpha \mathbf{v}_1 + \mathbf{v}_2$	$x - \alpha t$	u	$v(y)$

Table 4
Reduced equations corresponding to infinitesimal symmetries

operator	similarity reduced equations
\mathbf{v}_1	$v_y = 0$
\mathbf{v}_2	$3vv_y - 2v_y v_{y^2} - vv_{y^3} + 2v_y v_{y^4} + vv_{y^5} = 0$
\mathbf{v}_3	$-v + v_{y^2} - v_{y^4} + 3vv_y - 2v_y v_{y^2} - vv_{y^3} + 2v_y v_{y^4} + vv_{y^5} = 0$
$\alpha \mathbf{v}_1 + \mathbf{v}_2$	$-\alpha v_y + \alpha v_{y^3} - \alpha v_{y^5} + 3v v_y - 2v_y v_{y^2} - v v_{y^3} + 2v_y v_{y^4} + v v_{y^5} = 0$
$\alpha \mathbf{v}_1 + \mathbf{v}_3$	$-v - \alpha v_y + v_{y^2} + \alpha v_{y^3} - v_{y^4} - \alpha v_{y^5} + 3vv_y - 2v_y v_{y^2} - vv_{y^3} + 2v_y v_{y^4} + vv_{y^5} = 0$

where I is a smooth function of (x, t, u) ,

$$\frac{\partial I_1}{\partial x}, \quad \frac{\partial I_1}{\partial t}, \quad t \frac{\partial I_1}{\partial t} - u \frac{\partial I_1}{\partial u} - u_x \frac{\partial I_1}{\partial u_x} - 2u_t \frac{\partial I_1}{\partial u_t}, \quad (31)$$

where I_1 is a smooth function of (x, t, u, u_x, u_t) ,

$$\frac{\partial I_2}{\partial x}, \quad \frac{\partial I_2}{\partial t}, \quad t \frac{\partial I_2}{\partial t} - u \frac{\partial I_2}{\partial u} - u_x \frac{\partial I_2}{\partial u_x} - 2u_t \frac{\partial I_2}{\partial u_t} - u_{x^2} \frac{\partial I_2}{\partial u_{x^2}} - 2u_{xt} \frac{\partial I_2}{\partial u_{xt}} - 3u_{t^2} \frac{\partial I_2}{\partial u_{t^2}}, \quad (32)$$

where I_2 is a smooth function of $(x, t, u, u_x, u_t, u_{xx}, u_{xt}, u_{tt})$. The solutions of PDEs systems (30), (31) and (32) coming in table 5, where * and ** are refer to ordinary invariants and first order differential invariants respectively.

Table 5
differential invariants

vector field	ordinary invariant	1st order	2nd order
\mathbf{v}_1	t, u	$*, u_x, u_t$	$*, **, u_{xx}, u_{xt}, u_{tt}$
\mathbf{v}_2	x, u	$*, u_x, u_t$	$*, **, u_{xx}, u_{xt}, u_{tt}$
\mathbf{v}_3	$x, t u$	$*, t u_x, t^2 u_t$	$*, **, t u_{xx}, t^2 u_{xt}, t^3 u_{tt}$

7 Conservation laws for the higher order CH equation

Many methods for dealing with the conservation laws are derived, such as the method based on the Noether's theorem, the multiplier method, by the relationship between the conserved vector of a PDE and the Lie-Bcklund symmetry generators of the PDE, the direct method, etc.[22,23,24].

Now, we derive the conservation laws from the multiplier method.

Definition 8.1 A local conservation law of the PDE system (4) is a divergence expression

$$D_i \Phi^i[u] = D_1 \Phi^1[u] + \dots + D_n \Phi^n[u] = 0 \quad (33)$$

holding for all solutions of the system (4). In (33), $\Phi^i[u] = \Phi^i(x, u, \partial_u, \dots, \partial_u^r)$, $i = 1, \dots, n$, are called fluxes of the conservation law, and the highest-order derivative (r) present in the fluxes $\Phi^i[u]$ is called the order of a conservation law. [3]

In particular, a set of multipliers $\{\Lambda_\nu[U]\}_{\nu=1}^l = \{\Lambda_\nu(x, U, \partial_U, \dots, \partial_U^r)\}_{\nu=1}^l$ yields a divergence expression for the system $\Delta_\nu(x, u^{(n)})$ such that if the identity

$$\Lambda_\nu[U]\Delta_\nu[U] \equiv D_i \Phi^i[U] \quad (34)$$

holds identically for arbitrary functions $U(x)$. Then on the solutions $U(x) = u(x)$ of the system (4), if $\Lambda_\nu[U]$ is non-singular, one has local conservation law $\Lambda_\nu[u]\Delta_\nu[u] = D_i \Phi^i[u] = 0$.

Definition 8.2 The Euler operator with respect to U^j is the operator defined by

$$E_{U^j} = \frac{\partial}{\partial U^j} - D_i \frac{\partial}{\partial U^j} + \dots + (-1)^s D_{i_1} \dots D_{i_s} \frac{\partial}{\partial U_{i_1 \dots i_s}^j} + \dots \quad (35)$$

for $j = 1, \dots, q$. [3]

Theorem 8.3 The equations $E_{U^j} F(x, U, \partial_U, \dots, \partial_U^s) \equiv 0$, $j = 1, \dots, q$ hold for arbitrary $U(x)$ if and only if $F(x, U, \partial_U, \dots, \partial_U^s) \equiv D_i \Psi^i(x, U, \partial_U, \dots, \partial_U^{s-1})$ holds for some functions $\Psi^i(x, U, \partial_U, \dots, \partial_U^{s-1})$, $i = 1, \dots, q$. (Theorem 1.3.2, [3])

Theorem 8.4 A set of non-singular local multipliers $\{\Lambda_\nu(x, U, \partial_U, \dots, \partial_U^r)\}_{\nu=1}^l$ yields a local conservation law for the system $\Delta_\nu(x, u^{(n)})$ if and only if the set of identities

$$E_{U^j}(\Lambda_\nu(x, U, \partial_U, \dots, \partial_U^r) \Delta_\nu(x, u^{(n)})) \equiv 0, \quad j = 1, \dots, q, \quad (36)$$

holds for arbitrary functions $U(x)$. (Theorem 1.3.3, [3])

The set of equations (36) yields the set of linear determining equations to find all sets of local conservation law multipliers of the system (4). Now, we seek all local conservation law multipliers of the form $\Lambda = \xi(x, t, u)$ of the equation (3). The determining equations (36) become

$$E_U[\xi(x, t, U)(U_t - U_{x^2t} + U_{x^4t} + 3UU_x - 2U_xU_{x^2} - UU_{x^3} + 2U_xU_{x^4} + UU_{x^5})] \equiv 0, \quad (37)$$

where $U(x, t)$ are arbitrary function. Equation (37) split with respect to third order derivatives of U to yield the determining PDE system whose solutions are the sets of local multipliers of all nontrivial local conservation laws of the higher order CH equation.

The solution of the determining system (37) given by

$$\xi = c_1 U + c_2, \quad (38)$$

where c_1 and c_2 are arbitrary constants. So local multipliers given by

$$1) \xi = 1, \quad 2) \xi = U, \quad (39)$$

Each of the local multipliers ξ determines a nontrivial local conservation law $D_t \Psi + D_x \Phi = 0$ with the characteristic form

$$D_t \Psi + D_x \Phi \equiv \xi(U_t - U_{x^2t} + U_{x^4t} + 3UU_x - 2U_xU_{x^2} - UU_{x^3} + 2U_xU_{x^4} + UU_{x^5}), \quad (40)$$

To calculate the conserved quantities Ψ and Φ , we need to invert the total divergence operator. This requires the integration (by parts) of an expression in multi-dimensions involving arbitrary functions and its derivatives, which is a difficult and cumbersome task. The homotopy operator [25] is a powerful algorithmic tool (explicit formula) that originates from homological algebra and variational bi-complexes.

Definition 8.5 The 2-dimensional homotopy operator is a vector operator with two components, $\left(H_{u(x,t)}^{(x)}f, H_{u(x,t)}^{(t)}f\right)$, where

$$H_{u(x,t)}^{(x)}f = \int_0^1 \left(\sum_{j=1}^q I_{u^j}^{(x)}f \right) [\lambda u] \frac{d\lambda}{\lambda} \quad \text{and} \quad H_{u(x,t)}^{(t)}f = \int_0^1 \left(\sum_{j=1}^q I_{u^j}^{(t)}f \right) [\lambda u] \frac{d\lambda}{\lambda}. \quad (41)$$

The x-integrand, $I_{u^j}^{(x)}f$, is given by

$$I_{u^j}^{(x)}f = \sum_{k_1=1}^{M_1^j} \sum_{k_2=0}^{M_2^j} \left(\sum_{i_1=0}^{k_1-1} \sum_{i_2=0}^{k_2} B^{(x)} u_{x^{i_1} t^{i_2}}^j (-D_x)^{k_1-i_1-1} (-D_t)^{k_2-i_2} \right) \frac{\partial f}{\partial u_{x^{k_1} t^{k_2}}^j}, \quad (42)$$

where M_1^j, M_2^j are the order of f in u to x and t respectively and combinatorial coefficient

$$B^{(x)} = B(i_1, i_2, k_1, k_2) = \frac{\binom{i_1+i_2}{i_1} \binom{k_1+k_2-i_1-i_2-1}{k_1-i_1-1}}{\binom{k_1+k_2}{k_1}}.$$

Similarly, the t-integrand, $I_{u^j}^{(t)}f$, is defined as

$$I_{u^j}^{(t)}f = \sum_{k_1=0}^{M_1^j} \sum_{k_2=1}^{M_2^j} \left(\sum_{i_1=0}^{k_1} \sum_{i_2=0}^{k_2-1} B^{(t)} u_{x^{i_1} t^{i_2}}^j (-D_x)^{k_1-i_1} (-D_t)^{k_2-i_2-1} \right) \frac{\partial f}{\partial u_{x^{k_1} t^{k_2}}^j}, \quad (43)$$

where $B^{(t)}(i_2, i_1, k_2, k_1)$.

We apply homotopy operator to find conserved quantities Ψ and Φ which yield of multiplier $\xi = 1$. We have

$$f = U_t - U_{x^2}t + U_{x^4}t + 3UU_x - 2U_xU_{x^2} - UU_{x^3} + 2U_xU_{x^4} + UU_{x^5}, \quad (44)$$

the integrands (42) and (43) are

$$I_{u^j}^{(x)}f = 3u^2 - u_x^2 - \frac{2}{3}u_{xt} - 2uu_{x^2} + 2u_xu_{x^3} - u_{x^2}^2 + \frac{4}{5}u_{x^3}t + 2uu_{x^4}, \quad (45)$$

$$I_{u^j}^{(t)}f = u - \frac{1}{3}u_{x^2} + \frac{1}{5}u_{x^4},$$

apply (41) to the integrands (45), therefore

$$\Psi := H_{u(x,t)}^{(x)}f = \frac{3}{2}u^2 - \frac{1}{2}u_x^2 - \frac{2}{3}u_{xt} - uu_{x^2} + u_xu_{x^3} - \frac{1}{2}u_{x^2}^2 + \frac{4}{5}u_{x^3}t + uu_{x^4}, \quad (46)$$

$$\Phi := H_{u(x,t)}^{(t)}f = u - \frac{1}{3}u_{x^2} + \frac{1}{5}u_{x^4},$$

so, we have the first conservation law of the higher order CH equation respect to multiplier $\xi = 1$

$$D_x \left(\frac{3}{2}u^2 - \frac{1}{2}u_x^2 - \frac{2}{3}u_{xt} - uu_{x^2} + u_xu_{x^3} - \frac{1}{2}u_{x^2}^2 + \frac{4}{5}u_{x^3}t + uu_{x^4} \right) + D_t \left(u - \frac{1}{3}u_{x^2} + \frac{1}{5}u_{x^4} \right) = 0. \quad (47)$$

Now we find conservation law respect to multiplier $\xi = u$, in this case we have

$$f = U(U_t - U_{x^2t} + U_{x^4t} + 3UU_x - 2U_xU_{x^2} - UU_{x^3} + 2U_xU_{x^4} + UU_{x^5}), \quad (48)$$

the integrands are

$$\begin{aligned} I_{u^j}^{(x)} f &= 3u^3 + 3u^2u_{x^4} - \frac{4}{3}uu_{xt} + \frac{2}{3}u_xu_t - 3u^2u_{x^2} + \frac{8}{5}uu_{x^3t} - \frac{2}{5}u_{x^3}u_t - \frac{6}{5}u_xu_{x^2t} + \frac{4}{5}u_{x^2}u_{xt}, \\ I_{u^j}^{(t)} f &= u^2 - \frac{2}{3}uu_{x^2} + \frac{1}{3}u_x^2 + \frac{2}{5}uu_{x^4} - \frac{2}{5}u_xu_{x^3} + \frac{1}{5}u_{x^2}^2, \end{aligned} \quad (49)$$

applying 2-dimensional homotopy operator, we have

$$\begin{aligned} \Psi &:= H_{u(x,t)}^{(x)} f = u^3 + u^2u_{x^4} - \frac{2}{3}uu_{xt} + \frac{1}{3}u_xu_t - u^2u_{x^2} + \frac{4}{5}uu_{x^3t} - \frac{1}{5}u_{x^3}u_t - \frac{3}{5}u_xu_{x^2t} + \frac{2}{5}u_{x^2}u_{xt}, \\ \Phi &:= H_{u(x,t)}^{(t)} f = \frac{1}{2}u^2 - \frac{1}{3}uu_{x^2} + \frac{1}{6}u_x^2 + \frac{1}{5}uu_{x^4} - \frac{1}{5}u_xu_{x^3} + \frac{1}{10}u_{x^2}^2, \end{aligned} \quad (50)$$

so, the second conservation low of the higher order CH equation is

$$\begin{aligned} D_x(u^3 + u^2u_{x^4} - \frac{2}{3}uu_{xt} + \frac{1}{3}u_xu_t - u^2u_{x^2} + \frac{4}{5}uu_{x^3t} - \frac{1}{5}u_{x^3}u_t - \frac{3}{5}u_xu_{x^2t} + \frac{2}{5}u_{x^2}u_{xt}) \\ + D_t(\frac{1}{2}u^2 - \frac{1}{3}uu_{x^2} + \frac{1}{6}u_x^2 + \frac{1}{5}uu_{x^4} - \frac{1}{5}u_xu_{x^3} + \frac{1}{10}u_{x^2}^2) = 0. \end{aligned} \quad (51)$$

8 Conclusion

In this paper by applying the criterion of invariance of the equation under the infinitesimal prolonged infinitesimal generators, we find the most general Lie point symmetries group for higher order CH equation. By applying the nonclassical symmetry method for the higher order CH equation, we concluded that the analyzed model do not admit supplementary, nonclassical type symmetries. Also, we have constructed the optimal system of one-dimensional subalgebras for the higher order CH equation. The latter, creates the preliminary classification of group invariant solutions. The Lie invariants and similarity reduced equations corresponding to infinitesimal symmetries are obtained. We find the conservation laws from the multiplier method.

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